Day 27
How about a short cut? Or……
The Fundamental Theorem of Line Integrals

- Recap of Day 26- the Reader's Digest Condensed version.
- Consider the vector field: $\vec{F} = 2xy^4\hat{i} + 4x^2y^3\hat{j}$
    - Calculate $\int_C \vec{F} \cdot d\vec{r}$ for the parabola from (0,0) to (2,4).

You Try It :

Calculate $\int_C \vec{F} \cdot d\vec{r}$ for the line from (0,0) to (2,4).

- What's "special" about this vector field that makes it path independent?
- Observe that $\vec{F} = 2xy^4\hat{i} + 4x^2y^3\hat{j}$ is the ***gradient*** of the function $f(x,y) = x^2y^4$. Check it out! If you're wondering where the $f(x,y) = x^2y^4$ came from – stand by. At this point just recognize that, for this vector field, $\vec{F} = 2xy^4\hat{i} + 4x^2y^3\hat{j}$, $\vec{\nabla} f = \vec{F}$, where $f(x,y) = x^2y^4$. Another way to say this is $\vec{F} = \overline{grad}(x^2y^4)$. Therefore $\vec{F}$ is called a ***path independent*** vector field or a ***conservative*** vector field.

- **The Fundamental Theorem of Line Integrals:**

  If $\vec{F}$ is conservative then:

  $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(Q) - f(P)$, where $f(x,y)$ has gradient vector, $\vec{F}(x,y)$, and *C* is piecewise smooth curve and P is the starting point and Q the ending point of the curve *C.* The function, $f(x,y)$ is called the ***potential function*** for the vector field.

- Let's use the Fundamental Theorem of Line Integrals on our example:
    - Calculate $\int_C \vec{F} \cdot d\vec{r}$ for the line(or the parabola) from (0,0) to (2,4). $\vec{F} = 2xy^4\hat{i} + 4x^2y^3\hat{j}$.

- But where did the $f(x,y) = x^2y^4$ come from and how can I tell if the vector field is conservative(path independent)?
    - Example 1: Determine if $\vec{F}(x,y) = (2x+y)\hat{i} + (x+y^3)\hat{j}$ is conservative.
    - Example 2: Determine if $\vec{F}(x,y) = (x+y)\hat{i} + (xy)\hat{j}$ is conservative.

You Try It :

Determine if $\vec{F}(x,y) = 2x\ \hat{i} + 3y\ \hat{j}$ is conservative.

- So now we know how to tell if a vector field is conservative, if it is, how do we find the potential function $f(x,y)$?

- [Example 3](): Find the potential function $f(x,y)$ for the conservative vector field given in Example 1: $\vec{F}(x,y)=(2x+y)\hat{i}\ +(x+y^3)\hat{j}$.

You Try It:

Find the potential function $f(x,y)$ for the conservative vector field given in the *You Try It:* $\vec{F}(x,y)=2x\ \hat{i}+3y\ \hat{j}$. Answer given in Example 5 below.

- Using the Fundamental Theorem of Line Integrals
  - [Example 4:]() Calculate $\int_C \vec{F}\cdot d\vec{r}$ for the vector field in Example 1, $\vec{F}(x,y)=(2x+y)\hat{i}\ +(x+y^3)\hat{j}$ and the path C, given by the straight line going from the point (0,2) to the point (2,0).
  - [Example 5:]() Calculate $\int_C \vec{F}\cdot d\vec{r}$ for the vector field in the *You Try It,* $\vec{F}(x,y)=2x\ \hat{i}+3y\ \hat{j}$ and the path C, given by the unit square, centered at the origin, starting on the x-axis and going counter *clockwise.*

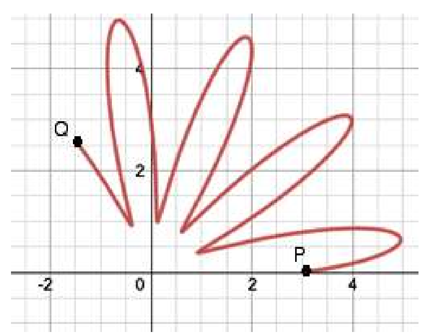
- If *C* is a closed path and $\vec{F}$ is conservative then:

$$\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{\nabla}f\cdot d\vec{r}=\int_P^P \frac{df}{dt}dt=f(P)-f(P)=0$$

You Try It:

Find $\int_C \vec{F}\cdot d\vec{r}$ for the vector field given by $\vec{F}(x,y)=2x\ \hat{i}+3y\ \hat{j}$ for the curve $C_1\!: y=2-2x^2$ from the point (0,2) to the point (1,0).

- Let's put it all together!
  - [Example 6:]() Calculate $\int_C \vec{F}\cdot d\vec{r}$ for the vector field $\vec{F}(x,y)=(x+y)\hat{i}\ +(x)\hat{j}$ and the path C, shown in the graph, from $P(3,0)$ to $Q\left(-\frac{3}{2},\frac{3\sqrt{3}}{2}\right)$.



You Try It:

Section 18.3 #45. Answer in Text.

- [Summary:]()
  - Test to see if the vector field is conservative.

- If it is, integrate to find the potential function and use the Fundamental Theorem of Line Integrals: f(endpoint) – f(initial point), where f is the potential function.
- If the vector field is not conservative, then you must use the "direct" method of calculation from Day 26.
- If the vector field is conservative, and if the path is closed, then $\oint_C \vec{F} \cdot d\vec{r} = 0$.