Day 26

Computing a Line Integral

- $\int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \vec{F}(x(t), y(t)) \cdot \vec{r}'(t)\ dt$, where $\vec{F}$ is the vector field and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is the parameterization of an oriented curve *C* and *a* and *b* are the endpoints of *C* in terms of t.
  - Because $\dfrac{d\vec{r}}{dt} = \vec{r}'(t)$ and solving for $d\vec{r}$, gives $d\vec{r} = \vec{r}'(t)\ dt$.
  - **Example 1:** If $\vec{F}(x,y) = (x+y)\hat{i} + (xy)\hat{j}$, find $\int_C \vec{F} \cdot d\vec{r}$ for the curve *C* given by $\vec{r}(t) = (2t)\hat{i} + (3t^2)\hat{j}$ f, for $0 \le t \le 1$.

Summary of Steps:

1) If $\vec{r}(t)$ is not given, find a parameterization for the curve *C.* If $\vec{r}(t)$ is given then proceed to step 2.
2) Use $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ to parameterize $\vec{F}$. That is write $\vec{F}$ in terms of t, where $x = x(t)$ and $y = y(t)$.
3) Find the derivative of $\vec{r}(t)$, which is $\vec{r}'(t)$.
4) Find the dot product of $\vec{F}$ and $\vec{r}'(t)$: $\vec{F} \cdot \vec{r}'(t)$.
5) Integrate the result from $t = a$ to $t = b$.

- **Example 2:** Now let's use the same vector field but change the path to the line that goes from the point (0,0) to the point (2,3). The end points are the same for this path as for the one above. Is the same amount of work done by the vector field over the 2 different paths?

You Try It 1:

Use the same vector field as above, but use the parameterization given by $\vec{r}(t) = t\ \hat{i} + \frac{3}{8}t^3\hat{j}$, for $0 \le t \le 2$ to find $\int_C \vec{F} \cdot d\vec{r}$ for this curve.

First use your calculator to sketch a graph of the path. Are the endpoints the same as above? Is the same amount of work done by the vector field over this path as the two paths above? Video Solution

- **Example 3:** Find $\int_C \vec{F} \cdot d\vec{r}$ for the vector field given by $\vec{F}(x,y) = 2x\ \hat{i} + 3y\ \hat{j}$ for the curve C_1: $y = 2 - 2x^2$ from the point (0,2) to the point (1,0).

You Try It 2:

Use the same vector field as above, $\vec{F}(x,y) = 2x\ \hat{i} + 3y\ \hat{j}$, but use the straight line path between the points from (0,2) to (1,0). Are the values for the line integrals the same over the two different paths?
Video Solution

- **Example 4:** Let's switch the orientation of the straight line path you did in the *You Try It* to go from (1,0) to (0,2). Call this path C_2.

  Calculate $\int_C \vec{F} \cdot d\vec{r}$ for this path. Use the same vector field:

  $\vec{F}(x,y) = 2x\ \hat{i} + 3y\ \hat{j}$.

  If we put the 2 paths together this would make a complete loop through the vector field. We'll call this new closed path C and it consists of $C_1 + C_2 = C$. Calculate the line integral along this curve by adding the answers to Example 3 and your answer above. Does this happen for all closed paths?

- **Example 5:** Go back to Example 1 and Example 2. Reorient one of the paths to form a closed path and calculate the line integral over that closed path. Use the vector field given in Example 1. Is it 0?

You Try It:

Use the path and field from Example 1 and reorient the path from the *You Try It* above to find the line integral over that closed path. Is it 0?

- So we have seen that:
  - For some vector fields we get the same value for the line integral no matter what the path is and for other vector fields the path does make a difference and we get different values for the line integral depending on the path we take.
  - If the vector field is one that the path doesn't matter – that is, it is a ***path independent*** vector field, then the line integral around a closed path is 0.
- Is there a way to determine which kind of vector field we have? Because, if we could, that would cut down on some substantial amount of work – especially on those closed path ones. The answer is YES!!! But, "Tomorrow is another day…." Imagine that with the Scarlett's southern drawl please.
- Everything we've done can be expanded to 3-space – just tack on a $z(t)$ and a $\hat{k}$.
- A different notation: *Differential Notation*
  - $\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y)\ dx + Q(x,y)\ dy$, where $\vec{F} = P(x,y)\hat{i} + Q(x,y)\hat{j}$ and dx and dy are the derivatives of x(t) and y(t) respectively.
  - Note this is still a dot product. It's just using some different symbols. So these two integrals are equivalent:

Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = 3xy\ \hat{i} + y^2 \hat{j}$ is the same as saying find $\int_C 3xy\ dx + y^2\ dy$

- **The Really Last Example:** Find $\int_C 3xy\ dx + y^2\ dy$ where C is the path $x = t^2$ and $y = t^3$ for $0 \le t \le 2$.