Day 14
Max/Mins in 3D

- Recall from Calculus I........
    - First Derivative Test
    - Second Derivative Test
- In Calculus III.......
    - At a local max/min the tangent plane must be horizontal.
        - This means z = constant
        - $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$
          So, $f_x(a,b) = f_y(a,b) = 0$
        - Therefore, points where $f_x(a,b) = f_y(a,b) = 0$ are called critical points.
    - Example: Given $f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 - 2xy + 5$, find the critical points.
    - How can we classify these critical points as max/min/saddle points?
- The 2nd Derivative Test in 3D
    - Suppose the point (a,b) is a critical point. Let $D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$
        - If D > 0 then the point (a,b) is either a mx or a min.
            - If $f_{xx} > 0$ at the point (a,b), then the point is a local minimum.
            - If $f_{xx} < 0$ at the point (a,b), then the point is a local maximum.
        - If D < 0 at the point (a,b), then the point is a saddle point.
        - If D = 0, then anything can happen: max, min, saddle point or none of these.
    - Example: Given $f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 - 2xy + 5$ and the critical points you found above, use the 2nd Derivative Test to classify the points as max/min/saddle point.
    - Contour Diagrams at critical points

✎ *You Try It*
Section 15.1 #9   Answer in Text.

- Did you just make up this test or what????? What is the explanation behind it? If you're interested, settle back.........it's a long story.
    - Recall the Taylor Series approximations from Calculus II
        - For a differentiable function f(x), the Taylor polynomial $P(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + ...$ is approximately equal to the function f(x) near x = 0.
    - Taylor Series approximations for functions of 2 variables

- The linear approximation near the point (0,0) is $L(x,y)=f(0,0)+f_x(0,0)x+f_y(0,0)y$. This is the tangent plane equation.
- The Quadratic approximation is

$$Q(x,y)=f(0,0)+f_x(0,0)x+\frac{1}{2!}f_{xx}(0,0)x^2+f_y(0,0)y+\frac{1}{2!}f_{yy}(0,0)y^2+f_{xy}(0,0)xy$$

- Let's assume that the critical point is at the origin (0,0,0). Doesn't change the shape. Then this simplifies to:
  $$Q(x,y)=\frac{1}{2}f_{xx}(0,0)x^2+\frac{1}{2}f_{yy}(0,0)y^2+f_{xy}(0,0)xy$$
- So any function can be approximated by a 3D quadratic, near its critical point by:
  $f(x,y)=ax^2+bxy+cy^2$, where a,b and c are constants.

- Completing the Square – on steroids!
  - $$ax^2+bxy+cy^2=a\left[\left(x+\frac{b}{2a}y\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)y^2\right]$$
    $$=a\left(x+\frac{b}{2a}y\right)^2+a\left(\frac{4ac-b^2}{4a^2}\right)y^2$$
  - The signs of the 2 squared terms are positive and therefore the shape is determined by the sign of *a* and $4ac-b^2$. Let's make $4ac-b^2=D$
- Classifying the Critical Point
  - If D > 0 and a > 0 then $+x^2+y^2$ and it has this shape $\cup$ and the critical point is a minimum.
  - If D > 0 and a < 0 then $-x^2-y^2$ and it has this shape $\cap$ and the critical point is a maximum.
  - If D < 0 and a > 0 then $+x^2-y^2$ and it's a saddle point.
  - If D < 0 and a < 0 then $-x^2+y^2$ and it's a saddle point.
    - This implies for the case of D < 0, the sign of *a* doesn't matter.
- Connection back to the 2nd Derivative test.
  - Recall, $f(x,y)\approx\frac{1}{2}f_{xx}(0,0)x^2+\frac{1}{2}f_{yy}(0,0)y^2+f_{xy}(0,0)xy$.
    $$f(x,y)\approx\frac{1}{2}f_{xx}(0,0)x^2+f_{xy}(0,0)xy+\frac{1}{2}f_{yy}(0,0)y^2$$
    Match the terms up: $ax^2+bxy+cy^2$
  - This implies
    $$D=4ac-b^2=4\left(\frac{1}{2}f_{xx}\right)\left(\frac{1}{2}f_{yy}\right)-\left(f_{xy}\right)^2$$
    $D=f_{xx}f_{yy}-\left(f_{xy}\right)^2$ and $a=f_{xx}$

This get's us to the 2^{nd} Derivative as stated above.