

Section 5.1 Two independent samples

Let  $X_1, X_2, \dots, X_n$  be a random sample of size n from population 1 and let  $Y_1, Y_2, \dots, Y_m$  be a random sample of size m from population 2.

Assume that the two samples are independent.

The purpose of this section is to determine whether there is a location shift between the two population distributions.

• The Mann-Whitney Test

Let F(x) be the distribution function of population distribution 1, and let G(x) be the distribution function of population distribution 2. In this section, it is assumed that F(x) and G(x) are identical (F(x) = G(x)) for all x or there exists a location shift between F(x) and G(x) (F(x) = G(x+c)) for all x.

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Statistical tests:
  Setting 1
    H_0: The two distributions are identical.
       (F(x) = G(x) \text{ for all } x)
    H_a: The distribution of X is shifted to the right of the distribution of Y.
       (P(X > Y) > P(Y > X))
  Setting 2
   H_0: The two distributions are identical.
       (F(x) = G(x) \text{ for all } x)
   H_a: The distribution of X is shifted to the left of the distribution of Y.
       (P(X > Y) < P(Y > X))
   Setting 3
    H_0: The two distributions are identical.
        (F(x) = G(x) \text{ for all } x)
    H_a: The distribution of X is shifted to the left or right of the distribution of Y.
        (P(X > Y) \neq P(Y > X))
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Let  $R(X_i)$  be the rank of  $X_i$  and let  $R(Y_j)$  be the rank of  $Y_j$  in the combined sample containing  $X_1, X_2, \dots, X_n$  and  $Y_1, Y_2, \dots, Y_m$ .

Test statistic:

Case A. There is no tie or there are only a few ties.

$$T = \sum_{i=1}^{n} R(X_i)$$

Case B. There are many ties.

$$T_{1} = \frac{T - \frac{n(N+1)}{2}}{\sqrt{\frac{nm}{N(N-1)} \sum_{i=1}^{N} R_{i}^{2} - \frac{nm(N+1)^{2}}{4(N-1)}}} \quad (N = n+m)$$

Caution:

For all the quiz problems and exam problems of this class, the effect of ties must be considered unless specified.

Setting 2

 $H_0$ : The two distributions are identical.

 $H_a$ : The distribution of X is shifted to the left of the distribution of Y.

Case A. There is no tie or there are only a few ties.

Reject  $H_0$  if  $T_{\text{(obs)}} < w_{\alpha}$ .

The value of  $w_{\alpha}$  can be found from Table A7.

Case B. There are many ties.

Reject  $H_0$  if  $T_{1 \text{(obs)}} < z_{\alpha}$ .

Large sample cases:

Case A. There is no tie or there are only a few ties.

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} < \frac{n(N+1)}{2} + z_{\alpha} \sqrt{\frac{nm(N+1)}{12}}$ .

$$p$$
-value =  $P\left(Z \le \frac{T - \frac{n(N+1)}{2} + 0.5}{\sqrt{\frac{nm(N+1)}{12}}}\right)$ 

Case B. There are many ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} < z_{\alpha}$ .

$$p$$
-value =  $P(Z \le T_{1(\text{obs})})$ 

Setting 1

 $H_0$ : The two distributions are identical.

 $H_a$ : The distribution of X is shifted to the right of the distribution of Y.

Case A. There is no tie or there are only a few ties.

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} > n(N+1) - w_{\alpha}$ .

Reject 
$$H_0$$
 if  $T'_{\text{(obs)}} = n(N+1) - T_{\text{(obs)}} < w_{\alpha}$ .

Case B. There are many ties.

Reject 
$$H_0$$
 if  $T_{1 \text{ (obs)}} > z_{1-\alpha}$ .

Large sample cases:

Case A. There is no tie or there are only a few ties.

$$\text{Reject } H_0 \text{ if } T_{\text{(obs)}} > \frac{n(N+1)}{2} + z_{1-\alpha} \sqrt{\frac{nm(N+1)}{12}} \ .$$

$$p$$
-value =  $P\left(Z \ge \frac{T - \frac{n(N+1)}{2} - 0.5}{\sqrt{\frac{nm(N+1)}{12}}}\right)$ 

Case B. There are many ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} > z_{1-\alpha}$ .

$$p$$
-value =  $P(Z \ge T_{1(\text{obs})})$ 

Setting 3

 $H_0$ : The two distributions are identical.

 $H_a$ : The distribution of X is shifted to the left or right of the distribution of Y.

Case A. There is no tie or there are only a few ties.

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} < w_{\alpha/2}$  or  $T_{\text{(obs)}} > n(N+1) - w_{\alpha/2}$ .

Case B. There are many ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} < z_{\alpha/2}$  or  $T_{1(\text{obs})} > z_{1-\alpha/2}$ .

Large sample cases:

Case A. There is no tie or there are only a few ties.

$$\text{Reject } H_0 \text{ if } T_{\text{(obs)}} < \frac{n(N+1)}{2} + z_{\alpha/2} \sqrt{\frac{nm(N+1)}{12}} \text{ or } T_{\text{(obs)}} > \frac{n(N+1)}{2} + z_{\text{1-}\alpha/2} \sqrt{\frac{nm(N+1)}{12}} \; .$$

$$p \text{-value} = 2 \min \left\{ P \left( Z \le \frac{T - \frac{n(N+1)}{2} + 0.5}{\sqrt{\frac{nm(N+1)}{12}}} \right), P \left( Z \ge \frac{T - \frac{n(N+1)}{2} - 0.5}{\sqrt{\frac{nm(N+1)}{12}}} \right) \right\}$$

Case B. There are many ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} < z_{\alpha/2}$  or  $T_{1(\text{obs})} > z_{1-\alpha/2}$ .

$$p \text{-value} = 2 \min \left\{ P \left( Z \leq T_{1 \text{(obs)}} \right), P \left( Z \geq T_{1 \text{(obs)}} \right) \right\}$$

### Example (Reaction Time)

An experimental psychologist wants to compare reaction times for individual males under the influence of drug A to those under the influence of drug B. Suppose the psychologist randomly assigns seven subjects to each of two groups, one group to receive drug A and the other to receive drug B. The reaction time for each subject is measured at the completion of the experiment. These data (with the exception of the measurement for one subject in group A who was eliminated from the experiment for personal reasons) are shown in the table.

Drug A	Drug B
1.96	2.11
2.24	2.43
1.71	2.07
2.41	2.71
1.62	2.50
1.93	2.84
	2.88

#### EXAMPLE 1

The senior class in a particular high school had 48 boys. Twelve boys lived on farms and the other 36 lived in town. A test was devised to see if farm boys in general were more physically fit than town boys. Each boy in the class was given a physical fitness test in which a low score indicates poor physical condition. The scores of the farm boys  $(X_i)$  and the town boys  $(Y_j)$  are as follows.

Farm Boys $(X)$	Town Boys $(Y)$
14.8 10.6	12.7 16.9 7.6 2.4 6.2 9.9
7.3 12.5	14.2 7.9 11.3 6.4 6.1 10.6
5.6 12.9	12.6 16.0 8.3 9.1 15.3 14.8
6.3 16.1	2.1 10.6 6.7 6.7 10.6 5.0
9.0 11.4	17.7 5.6 3.6 18.6 1.8 2.6
4.2 2.7	11.8 5.6 1.0 3.2 5.9 4.0

Do the data provide evidence to show that farm boys tend to be more fit than town boys?

#### EXAMPLE 2

A simple experiment was designed to see if flint in area A tended to have the same degree of hardness as flint in area B. Four sample pieces of flint were collected in area A and five sample pieces of flint were collected in area B. To determine which of two pieces of flint was harder, the two pieces were rubbed against each other. The piece sustaining less damage was judged the harder of the two. In this manner all nine pieces of flint were ordered according to hardness. The rank 1 was assigned to the softest piece, rank 2 to the next softest, and so on.

Origin of Piece	Rank
A	1
A	2
A	3
B	4
A	5
B	6
B	7
B	8
B	9

• Confidence Interval for the Difference between Two Medians (Means)

Let  $X_1, X_2, \dots, X_n$  be a random sample of size n from population 1 and let  $Y_1, Y_2, \dots, Y_m$  be a random sample of size m from population 2. Assume that the two samples are independent.

Let F(x) be the distribution function of population distribution 1, and let G(x) be the distribution function of population distribution 2. It is assumed that F(x) and G(x) are identical (F(x) = G(x) for all x) or there exists a location shift between F(x) and G(x) (F(x) = G(x+c) for all x).

The purpose is to construct a  $1-\alpha$  confidence interval for M(X)-M(Y) (or E(X)-E(Y)).

Find the value 
$$k = w_{\alpha/2} - \frac{n(n+1)}{2}$$
 (or  $k = z_{\alpha/2} - \frac{n(n+1)}{2}$  for large sample).

The lower confidence limit L is the kth smallest difference  $X_i - Y_j$ .

The upper confidence limit U is the kth largest difference  $X_i - Y_j$ .

## EXAMPLE 3

A cake batter is to be mixed until it reaches a specified level of consistency. Five batches of the batter are mixed using mixer A, and another five batches are mixed using mixer B. The required times for mixing are given as follows (in minutes).

Mixer A	Mixer B
7.3	7.4
6.9	6.8
7.2	6.9
7.8	6.7
7.2	7.1

Find a 95% confidence interval for the mean difference in mixing time.

Section 5.2 Several independent samples

 $X_{11}, X_{12}, \dots, X_{1n_1}$ : a random sample from population 1

 $X_{21}, X_{22}, \cdots, X_{2n_2}$ : a random sample from population 2

...

 $X_{k1}, X_{k2}, \dots, X_{kn_k}$ : a random sample from population k

Assume that all the samples are independent. The purpose of this section is to determine whether there is a location shift among the population distributions.

• The Kruskal-Wallis Test

 $H_0$ : All the k population distributions are identical.

 $H_a$ : At least one of the population distributions tends to yield larger observations than at least one of the other population distributions.

 $H_0$ : All the k population distributions are identical.

 $H_a$ : At least two population distributions have different medians (means).

Test statistic:

$$T = \frac{1}{S^2} \left( \sum_{i=1}^{k} \frac{R_i^2}{n_i} - \frac{N(N+1)^2}{4} \right) \quad \left( S^2 = \frac{1}{N-1} \left( \sum_{i=1}^{k} \sum_{j=1}^{n_i} R^2 \left( X_{ij} \right) - \frac{N(N+1)^2}{4} \right) \right)$$

If there are no ties, then  $T = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(N+1)$ .

Reject  $H_0$  if  $T_{\text{(obs)}} > w_{1-\alpha}$ .

$$p$$
-value =  $P(T \ge T_{\text{(obs)}})$ 

The value of  $w_{1-\alpha}$  can be found from Table A8.  $(k=3; n_1, n_2, n_3 \le 5)$ 

If no table is available, reject  $H_0$  if  $T_{\text{(obs)}} > \chi^2_{k-1, 1-\alpha}$ .

## **Multiple Comparisons**

If the null hypothesis is rejected, then the following procedure is needed to determine which pairs of populations tend to differ.

It can be concluded that population i and population j seem to be different if

$$\left| \; \frac{R_i}{n_i} - \frac{R_j}{n_j} \; \right| > t_{N-k,1-\alpha/2} \sqrt{\left(\frac{N-1-T}{N-k}S^2\right) \left(\frac{1}{n_i} + \frac{1}{n_j}\right)} \; .$$

Here  $T = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(N+1)$  if there are no ties; otherwise

$$T = \frac{1}{S^{2}} \left( \sum_{i=1}^{k} \frac{R_{i}^{2}}{n_{i}} - \frac{N(N+1)^{2}}{4} \right).$$

$$\left(S^{2} = \frac{1}{N-1} \left( \sum_{i=1}^{k} \sum_{j=1}^{n_{i}} R^{2} \left(X_{ij}\right) - \frac{N(N+1)^{2}}{4} \right) \right)$$

#### EXAMPLE 1 (Data from Example 4.3.1)

Four different methods of growing corn were randomly assigned to a large number of different plots of land and the yield per acre was recorded for each plot.

Plot 1	Plot 2	Plot 3	Plot 4
83	91	101	78
91	90	100	82
94	81	91	81
89	83	93	77
89	84	96	79
96	83	95	81
91	88	94	80
92	91		81
90	89		
	84		

Use the Kruskal-Wallis test to check whether there is significant difference among four methods.

#### EXAMPLE (Performance Rating)

A company hires employees for its management staff from three local colleges. Recently, the company's personnel department has been collecting and reviewing annual performance ratings in an attempt to determine whether there are differences in performance among the managers hired from three colleges. Performance rating data are available from independent samples of seven employees from college A, six employees from college B, and seven employees from college C. The data are shown in the table, where the overall performance rating of each manager is given on a 0-100 scale, with 100 being the best.

College A	College B	College C
25	60	50
70	20	70
60	30	60
85	15	80
95	40	90
90	35	70
80		75

Do the data provide enough evidence to conclude that there is a difference among three colleges? Use  $\alpha$ =0.05.

In the case that there are many ties, simplified formulas can be used for calculation.

	Population 1	Population 2		Population k	
Category 1	$O_{11}$	$O_{12}$		$O_{1k}$	
Category 2	$O_{21}$	$O_{22}$		$O_{2k}$	
				•••	
Category c	$O_{c1}$	$O_{c2}$	•••	$O_{ck}$	
	$n_1$	$n_2$		$n_k$	

In the case that there are many ties, simplified formulas can be used for calculation.

	Population 1	Population 2	 Population k	Row Total
Category 1	$O_{11}$	$O_{12}$	 $O_{1k}$	$t_1$
Category 2	$O_{21}$	$O_{22}$	 $O_{2k}$	$t_2$
			 •••	
Category c	$O_{c1}$	$O_{c2}$	 $O_{ck}$	$t_c$
	$n_1$	$n_2$	 $n_k$	N

$$\begin{array}{ccc} \text{Row} & \text{Average} \\ \text{Total} & Rank \\ t_1 & \overline{R}_1 \\ t_2 & \overline{R}_2 \\ \cdots \\ t_c & \overline{R}_c \\ N \end{array}$$

$$\overline{R}_{1} = \frac{1+2+\dots+t_{1}}{t_{1}} = \frac{t_{1}+1}{2}$$

$$\overline{R}_{2} = \frac{(t_{1}+1)+(t_{1}+2)+\dots+(t_{1}+t_{2})}{t_{2}} = t_{1} + \frac{t_{2}+1}{2}$$

$$\overline{R}_{3} = \frac{((t_{1}+t_{2})+1)+((t_{1}+t_{2})+2)+\dots+((t_{1}+t_{2})+t_{3})}{t_{3}} = (t_{1}+t_{2}) + \frac{t_{3}+1}{2}$$

$$\overline{R}_c = \sum_{i=1}^{c-1} t_i + \frac{t_c + 1}{2} = \left(t_1 + t_2 + \dots + t_{c-1}\right) + \frac{t_c + 1}{2}$$

$$R_j = \sum_{i=1}^c O_{ij} \overline{R}$$

$$S^{2} = \frac{1}{N-1} \left[ \sum_{i=1}^{c} t_{i} \overline{R}_{i}^{2} - \frac{N(N+1)^{2}}{4} \right]$$

$$T = \frac{1}{S^{2}} \left( \sum_{i=1}^{k} \frac{R_{i}^{2}}{n_{i}} - \frac{N(N+1)^{2}}{4} \right)$$

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} > \chi^2_{k-1,1-\alpha}$ .

## EXAMPLE 2

Three instructors compared the grades they assigned over the past semester to see if some of them tended to give lower grades than others.

 $H_{\rm 0}$ : The three instructors have the same grade distributions.

 $H_a$ : Some instructors tend to grade lower than others.

Grades	Instructor 1	Instructor 2	Instructor 3	
A	4	10	6	
В	14	6	7	
C	17	9	8	
D	6	7	6	
F	2	6	1	

Section 5.3 A test for equal variances

 $X_1, X_2, \dots, X_n$ : a random sample from population 1

 $Y_1, Y_2, \dots, Y_m$ : a random sample from population 2

Assume that the two samples are independent. The purpose is to determine whether there exists difference between the two population variances.

• The Squared Ranks Test for Variances

Setting 1

 $H_0$ : Two population distributions are identical except location shift.

$$H_a: Var(X) > Var(Y)$$

Setting 2

 $H_0$ : Two population distributions are identical except location shift.

$$H_a: Var(X) < Var(Y)$$

Setting 3

 $H_0$ : Two population distributions are identical except location shift.

$$H_a: Var(X) \neq Var(Y)$$

$$U_i = |X_i - \mu_1| \quad (i = 1, 2, \dots, n)$$

$$V_j = |Y_j - \mu_2| \quad (j = 1, 2, \dots, m)$$

If  $\mu_1$  and  $\mu_2$  are unknown, use  $U_i = \left| \ X_i - \overline{X} \ \right|$  and  $V_j = \left| \ Y_j - \overline{Y} \ \right|$ .

Rank  $U_1, U_2, \cdots, U_n, V_1, V_2, \cdots, V_m$ .

Let  $R(U_i)$  be the rank of  $U_i$ , and Let  $R(V_j)$  be the rank of  $V_j$ .

Test statistic:

Case A. There are no ties.

$$T = \sum_{i=1}^{n} \left[ R(U_i) \right]^2$$

Case B. There are ties.

$$T_{1} = \frac{T - n \overline{R^{2}}}{\left[\frac{nm}{N(N-1)} \sum_{i=1}^{N} R_{i}^{4} - \frac{nm}{N-1} (\overline{R^{2}})^{2}\right]^{1/2}}$$

$$\begin{pmatrix}
N = n + m \\
\overline{R^2} = \frac{\sum_{i=1}^{n} \left[ R(U_i) \right]^2 + \sum_{j=1}^{m} \left[ R(V_j) \right]^2}{N} \\
\sum_{i=1}^{N} R_i^4 = \sum_{i=1}^{n} \left[ R(U_i) \right]^4 + \sum_{j=1}^{m} \left[ R(V_j) \right]^4
\end{pmatrix}$$

Setting 1

 $H_0$ : Two population distributions are identical except location shift.

$$H_a: Var(X) > Var(Y)$$

Case A. There are no ties.

Reject  $H_0$  if  $T_{\text{(obs)}} > w_{1-\alpha}$  (Table A9 for  $m \le 10, n \le 10$ ).

$$w_p = \frac{n(N+1)(2N+1)}{6} + z_p \sqrt{\frac{mn(N+1)(2N+1)(8N+1)}{180}} \quad {m>10 \choose n>10}$$

$$p$$
-value =  $P(T \ge T_{\text{(obs)}})$ 

Case B. There are ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} > z_{1-\alpha}$  (Table A1)

$$p$$
-value =  $P(T_1 \ge T_{1(\text{obs})})$ 

Setting 2

 $H_0$ : Two population distributions are identical except location shift.

$$H_a$$
:  $Var(X) < Var(Y)$ 

Case A. There are no ties.

Reject  $H_0$  if  $T_{\text{(obs)}} < w_{\alpha}$  (Table A9 for  $m \le 10, n \le 10$ ).

$$w_p = \frac{n(N+1)(2N+1)}{6} + z_p \sqrt{\frac{mn(N+1)(2N+1)(8N+1)}{180}} \quad \binom{m>10}{n>10}$$

$$p$$
-value =  $P(T \le T_{\text{(obs)}})$ 

Case B. There are ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} < z_{\alpha}$  (Table A1).

$$p$$
-value =  $P(T_1 \le T_{1(\text{obs})})$ 

Setting 3

 $H_0$ : Two population distributions are identical except location shift.

$$H_a: Var(X) \neq Var(Y)$$

Case A. There are no ties.

Reject  $H_0$  if  $T_{\text{(obs)}} < w_{\alpha/2}$  or  $T_{\text{(obs)}} > w_{1-\alpha/2}$  (Table A9 for  $m \le 10, n \le 10$ ).

$$w_p = \frac{n(N+1)(2N+1)}{6} + z_p \sqrt{\frac{mn(N+1)(2N+1)(8N+1)}{180}} \quad {m>10 \choose n>10}$$

$$p$$
-value =  $2 \min \left\{ P\left(T \le T_{\text{(obs)}}\right), P\left(T \ge T_{\text{(obs)}}\right) \right\}$ 

Case B. There are ties.

Reject 
$$H_0$$
 if  $T_{1(\text{obs})} < z_{\alpha/2}$  or  $T_{1(\text{obs})} > z_{1-\alpha/2}$  (Table A1)

$$p\text{-value} = 2\min\left\{P\left(T_1 \le T_{1(\text{obs})}\right), P\left(T_1 \ge T_{1(\text{obs})}\right)\right\}$$

#### EXAMPLE 1

A food packaging company would like to be reasonably sure that the boxes of cereal it produces do in fact contain at least the number of ounces of cereal stamped on the outside of the box. In order to do this it must set the average amount per box a little above the advertised amount, because the unavoidable variation caused by the packaging machine will sometimes put a little less or a little more cereal in the box. A machine with smaller variation would save the company money because the average amount per box could be adjusted to be closer to the advertised amount. A new machine is being tested to see if it is less variable than the present machine, in which case it will be purchased to replace the old machine. Several boxes are filled with cereal using the present machine and the amount in each box is measured.

Original Measurement		Absolute Deviation		Rank		Squared Rank	
Present (X)	New (Y)	Present (U)	New (V)	Present	New	Present	New
10.8	10.8	0.06	0.01	4	2	16	4
11.1	10.5	0.36	0.29	10	8	100	64
10.4	11.0	0.34	0.21	9	7	81	49
10.1	10.9	0.64	0.11	12	6	144	36
11.3	10.8	0.56	0.01	11	2	121	4
	10.7		0.09		5		25
	10.8		0.01		2		4
$\bar{X} = 10.74$	$\bar{Y} = 10.79$					T = 462	

A Test For More Than Two Samples

 $X_{11}, X_{12}, \dots, X_{1n_1}$ : a random sample from population 1

 $X_{21}, X_{22}, \cdots, X_{2n_2}$ : a random sample from population 2

 $X_{k1}, X_{k2}, \dots, X_{kn_k}$ : a random sample from population k

Assume that all k samples are independent. The purpose is to determine whether there exists difference among the population variances.

Statistical Test

 $H_0$ : All k population distributions are identical except location shift.

 $H_a$ : At least two population variances are different.

$$U_{ij} = |X_{ij} - \mu_i| \quad (j = 1, 2, \dots, n_i; i = 1, 2, \dots, k)$$

If  $\mu_{\!\scriptscriptstyle 1},\,\mu_{\!\scriptscriptstyle 2},\cdots,\,\mu_{\!\scriptscriptstyle k}$  are unknown, replace them by  $\,\overline{\!X}_{\!\scriptscriptstyle 1},\,\overline{\!X}_{\!\scriptscriptstyle 2},\cdots,\,\overline{\!X}_{\!\scriptscriptstyle k}$  .

Rank 
$$U_{ii}$$
  $(j=1,2,\dots,n_i; i=1,2,\dots,k)$ .

Let  $R(U_{ij})$  be the rank of  $U_{ij}$   $(j=1,2,\cdots,n_i;i=1,2,\cdots,k)$ .

Define 
$$S_i = \sum_{i=1}^{n_i} \left[ R(U_{ij}) \right]^2 (i=1,2,\cdots,k)$$
.

est statistic 
$$T_2 = \frac{1}{D^2} \left[ \sum_{i=1}^k \frac{S_i^2}{n_i} - N(\overline{S})^2 \right]$$
 
$$\left[ N = \sum_{i=1}^k n_i \right]$$
 
$$\overline{S} = \frac{1}{N} \sum_{i=1}^k S_i$$
 
$$D^2 = \frac{1}{N-1} \left[ \sum_{i=1}^k \sum_{j=1}^{n_i} \left( R(U_{ij}) \right)^4 - N(\overline{S})^2 \right]$$
 eject  $H_0$  if  $T_{2(\text{obs})} > \chi^2_{k-1,1-\alpha}$ .

Reject  $H_0$  if  $T_{2(\text{obs})} > \chi^2_{k-1,1-\alpha}$ .

$$p$$
-value =  $P(T_2 \ge T_{2(\text{obs})})$ 

Section 5.4 Measure of rank correlation

Let  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  be a random sample from a bivariate probability distribution.

The purpose of this section is to discuss the correlation between *X* and *Y*.

• Pearson's Coefficient of Correlation

$$r = \frac{\sum_{i=1}^{n} (X_{i} - \overline{X})(Y_{i} - \overline{Y})}{\sqrt{\sum_{i=1}^{n} (X_{i} - \overline{X})^{2} \sum_{i=1}^{n} (Y_{i} - \overline{Y})^{2}}}$$

$$r = \frac{\sum_{i=1}^{n} X_{i} Y_{i} - n \overline{X} \overline{Y}}{\sqrt{\left(\sum_{i=1}^{n} X_{i}^{2} - n(\overline{X})^{2}\right) \left(\sum_{i=1}^{n} Y_{i}^{2} - n(\overline{Y})^{2}\right)}}$$

• Spearman's Rho  $(\rho)$ 

$$\rho = \frac{\sum_{i=1}^{n} R(X_i) R(Y_i) - n \left(\frac{n+1}{2}\right)^2}{\sqrt{\left(\sum_{i=1}^{n} (R(X_i))^2 - n \left(\frac{n+1}{2}\right)^2\right) \left(\sum_{i=1}^{n} (R(Y_i))^2 - n \left(\frac{n+1}{2}\right)^2\right)}}$$

In the case that there are no ties, the formula can be simplified as

$$\rho = 1 - \frac{6\sum_{i=1}^{n} \left[R(X_i) - R(Y_i)\right]^2}{n(n^2 - 1)} = 1 - \frac{6T}{n(n^2 - 1)}.$$

## EXAMPLE 1

Twelve MBA graduates are studied to measure the strength of the relationship between their score on the GMAT, which they took prior to entering graduate school, and their grade point average while they were in the MBA program. Their GMAT scores and their GPAs are given below.

Student	$GMAT\left(X\right)$	GPA(y)
1	710	4.0
2	610	4.0
3	640	3.9
4	580	3.8
5	545	3.7
6	560	3.6
7	610	3.5
8	530	3.5
9	560	3.5
10	540	3.3
11	570	3.2
12	560	3.2

Find the Spearman's correlation coefficient.

A. Use the formula considering ties.

X	Y	R(X)	R(Y)	$R^2(X)$	$R^2(Y)$	R(X)R(Y)
710	4.0	12	11.5	144	132.25	138
610	4.0	9.5	11.5	90.25	132.25	109.25
640	3.9	11	10	121	100	110
580	3.8	8	9	64	81	72
545	3.7	3	8	9	64	24
560	3.6	5	7	25	49	35
610	3.5	9.5	5	90.25	25	47.5
530	3.5	1	5	1	25	5
560	3.5	5	5	25	25	25
540	3.3	2	3	4	9	6
570	3.2	7	1.5	49	2.25	10.5
560	3.2	5	1.5	25	2.25	7.5
		78	78	647.5	647	589.75

B. Use the formula ignoring ties.

GMAT 
$$(X)$$
 GPA  $(Y)$   $R(X)$   $R(Y)$   $[R(X)-R(Y)]^2$ 

710 4.0 12 11.5 0.25
610 4.0 9.5 11.5 4
640 3.9 11 10 1
580 3.8 8 9 1
545 3.7 3 8 25
560 3.6 5 7 4
610 3.5 9.5 5 20.25
530 3.5 1 5 16
560 3.5 5 5 0
540 3.3 2 3 1
570 3.2 7 1.5 30.25
560 3.2 5 1.5 12.25

```
Statistical Tests

Test statistic: \rho

Setting 1

H_0: There is no correlation between X and Y.

H_a: X and Y are positively correlated.

Reject H_0 if \rho_{\text{(obs)}} > w_{1-\alpha}. \left(w_{1-\alpha} \text{ can be found from Table A10}\right)

\left(n \le 30 \text{ and no ties}\right)

Reject H_0 if \rho_{\text{(obs)}} > \frac{z_{1-\alpha}}{\sqrt{n-1}}.

\left(n > 30 \text{ or many ties}\right)

p\text{-value} = P\left(\rho \ge \rho_{\text{(obs)}}\right) = P\left(Z \ge \rho_{\text{(obs)}}\sqrt{n-1}\right)
```

```
Setting 2 H_0: \text{ There is no correlation between } X \text{ and } Y. H_a: X \text{ and } Y \text{ are negatively correlated.} \text{Reject } H_0 \text{ if } \rho_{\text{(obs)}} < -w_{\text{l-}\alpha}. \quad \left(w_{\text{l-}\alpha} \text{ can be found from Table A10}\right) \left(n \leq 30 \text{ and no ties}\right) \text{Reject } H_0 \text{ if } \rho_{\text{(obs)}} < \frac{z_\alpha}{\sqrt{n-1}}. \left(n > 30 \text{ or many ties}\right) p\text{-value} = P\left(\rho \leq \rho_{\text{(obs)}}\right) = P\left(Z \leq \rho_{\text{(obs)}}\sqrt{n-1}\right)
```

#### Setting 3

 $H_0$ : There is no correlation between X and Y.

 $H_a$ : X and Y are positively or negatively correlated.

Reject 
$$H_0$$
 if  $|\rho_{\text{(obs)}}| > w_{1-\alpha/2}$ .  $(w_{1-\alpha} \text{ can be found from Table A10})$   $(n \le 30 \text{ and no ties})$ 

Reject 
$$H_0$$
 if  $|\rho_{\text{(obs)}}| > \frac{z_{1-\alpha/2}}{\sqrt{n-1}}$ .  
 $(n > 30 \text{ or many ties})$ 

$$p \text{-value} = 2 \min \left\{ P\left(\rho \le \rho_{\text{(obs)}}\right), \ P\left(\rho \ge \rho_{\text{(obs)}}\right) \right\} = 2 \ P\left(Z \ge \left| \ \rho_{\text{(obs)}} \ \left| \sqrt{n-1} \right. \right) \right\}$$

## EXAMPLE 2 (Example 1 continued)

Suppose the twelve MBA graduates in Example 1 are a random sample of all recent MBA graduates, and we want to know if there is a tendency for high GPAs to be associated with high GMAT scores.

 $H_0$ : GPAs are independent of GMAT scores.

 $H_a$ : High GPAs tend to be associated with high GMAT scores,

• Kendall's Tau  $(\tau)$ 

$$(X_i, Y_i)$$
 and  $(X_j, Y_j)$  are said to be concordant if  $\frac{Y_i - Y_j}{X_i - X_j} > 0$ .

Let  $N_c$  be the total number of concordant pairs.

$$\left(X_i,Y_i\right)$$
 and  $\left(X_j,Y_j\right)$  are said to be discordant if  $\frac{Y_i-Y_j}{X_i-X_j}$  < 0 .

Let  $N_d$  be the total number of discordant pairs.

Kendall's 
$$\tau$$
:  $\tau = \frac{N_c - N_d}{n(n-1)/2}$  (no ties)

Considering ties:

 $\left(X_{i},Y_{i}\right)$  and  $\left(X_{j},Y_{j}\right)$  are said to be half concordant and half discordant if

$$Y_i = Y_j \ (X_i \neq X_j).$$

No comparison is made if  $X_i = X_j$ .

Kendall's  $\tau$ :  $\tau = \frac{N_c - N_d}{N_c + N_d}$  (considering ties)

EXAMPLE 3 (Example 1 continued)

Arrangement of the data  $(X_i, Y_i)$  according to increasing values of X.

	$X_i, Y_i$	Concordant Pairs Below (X <sub>i</sub> , Y <sub>i</sub> )	Discordant Pairs Below (X <sub>i</sub> , Y <sub>i</sub> )
	(530, 3.5)	7	4
	(540, 3.3)	8	2
	(545, 3.7)	4	5
	((560, 3.2)	5.5	0.5
tie	(560, 3.5)	4.5	1.5
	(560, 3.6)	4	2
	(570, 3.2)	5	0
	(580, 3.8)	3	1
tie	((610, 3.5)	2	0
ne	(610, 4.0)	0.5	1.5
	(640, 3.9)	1	0
	(740, 4.0)		
		$N_c = 44.5$	$N_d = 17.5$

Statistical Tests

Test statistic:  $T = N_c - N_d$  or  $\tau$ 

The quantiles for T and  $\tau$  are available in Table A11.

Setting 1

 $H_0$ : There is no correlation between X and Y.

 $H_a$ : X and Y are positively correlated.

Reject  $H_0$  if  $T_{({\rm obs})} > w_{{\rm l}-\alpha}$ . ( $w_{{\rm l}-\alpha}$  can be found from Table A11)

 $(n \le 60 \text{ and no ties})$ 

Reject  $H_0$  if  $T_{\text{(obs)}} > z_{1-\alpha} \sqrt{\frac{n(n-1)(2n+5)}{18}}$ .

(n > 60 or many ties)

$$p$$
-value =  $P(T \ge T_{\text{(obs)}}) = P\left(Z \ge \frac{(T_{\text{(obs)}} - 1)\sqrt{18}}{\sqrt{n(n-1)(2n+5)}}\right)$ 

### Setting 2

 $H_0$ : There is no correlation between X and Y.

 $H_a$ : X and Y are negatively correlated.

Reject  $H_0$  if  $T_{\text{(obs)}} < -w_{1-\alpha}$ . ( $w_{1-\alpha}$  can be found from Table A11) ( $n \le 60$  and no ties)

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} < z_{\alpha} \sqrt{\frac{n(n-1)(2n+5)}{18}}$ .  
 $(n > 60 \text{ or many ties})$ 

$$p$$
-value =  $P(T \le T_{\text{(obs)}}) = P\left(Z \le \frac{(T_{\text{(obs)}} + 1)\sqrt{18}}{\sqrt{n(n-1)(2n+5)}}\right)$ 

#### Setting 3

 $H_0$ : There is no correlation between X and Y.

 $H_a$ : X and Y are positively or negatively correlated.

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} < -w_{1-\alpha/2}$  or  $T_{\text{(obs)}} > w_{1-\alpha/2}$ . ( $n \le 60$  and no ties)

$$\begin{array}{c} \text{Reject } H_0 \text{ if } T_{\text{(obs)}} < z_{\alpha/2} \sqrt{\frac{n(n-1)(2n+5)}{18}} \text{ or } T_{\text{(obs)}} > z_{1-\alpha/2} \sqrt{\frac{n(n-1)(2n+5)}{18}} \,. \\ & \left( n > 60 \text{ or many ties} \right) \end{array}$$

$$p$$
-value =  $2 \min \left\{ P\left(T \le T_{\text{(obs)}}\right), P\left(T \ge T_{\text{(obs)}}\right) \right\}$ 

$$= 2 \min \left\{ P \left( Z \le \frac{\left( T_{(\text{obs})} + 1 \right) \sqrt{18}}{\sqrt{n(n-1)(2n+5)}} \right), P \left( Z \ge \frac{\left( T_{(\text{obs})} - 1 \right) \sqrt{18}}{\sqrt{n(n-1)(2n+5)}} \right) \right\}$$

EXAMPLE 4 (Example 3 continued)

 $H_0$ : GPAs are independent of GMAT scores.

 $H_a$ : High GPAs tend to be associated with high GMAT scores,

## • The Daniels Test for Trend

## EXAMPLE 5

In Example 3.5.2, nineteen years of annual precipitation records are given.

	Precipitation Yi	
Year Xi	(inches)	
1950	45.25	
1951	45.83	
1952	41.77	
1953	36.26	
1954	45.27	
1955	52.25	The Cox-Stuart test for trend test
1956	35.37	was used in Example 3.5.2.
1957	57.16	was used in Example 3.5.2.
1958	35.37	
1959	58.32	
1960	41.05	
1961	33.72	
1962	45.73	
1963	37.90	
1964	41.72	
1965	36.07	
1966	49.83	
1967	36.24	
1968	39.90	

Test to see if there is a trend (downward or upward) in the data.

Year X <sub>i</sub>	Precipitation $Y_i$ (inches)	$R(X_i)$	$R(Y_i)$	$[R(X_i) - R(Y_i)]^2$
1950	45.25	1	12	121
1951	45.83	2	15	169
1952	41.77	3	11	64
1953	36.26	4	6	4
1954	45.27	5	13	64
1955	52.25	6	17	121
1956	35.37	7	2.5	20.25
1957	57.16	8	18	100
1958	35.37	9	2.5	42.25
1959	58.32	10	19	81
1960	41.05	11	9	4
1961	33.72	12	1	121
1962	45.73	13	14	1
1963	37.90	14	7	49
1964	41.72	15	10	25
1965	36.07	16	4	144
1966	49.83	17	16	1
1967	36.24	18	5	169
1968	39.90	19	8	121
				Total 1421.5

Section 5.7 The one-sample or matched-pairs case

Let  $(X_1, Y_1), (X_2, Y_2), \dots, (X_{n'}, Y_{n'})$  be a random sample from a bivariate population distribution (X, Y). The purpose of this section is to compare the medians of X and Y.

• Wilcoxon Signed Ranks Test

Define 
$$D_i = Y_i - X_i \ (i = 1, 2, \dots, n')$$
.

Eliminate the pairs with zero difference. Let n be the number of pairs with nonzero difference.

Assumptions:

- 1.  $D_1, D_2, \dots, D_n$  are independent.
- 2. The distributions of  $D_1, D_2, \dots, D_n$  are symmetric.
- 3.  $D_1, D_2, \dots, D_n$  have the same median.

Statistical Tests

Setting 1

$$H_0: M(Y) \le M(X) \quad (M(D) \le 0)$$

$$H_a: M(Y) > M(X) \quad (M(D) > 0)$$

Setting 2

$$H_0: M(Y) \ge M(X) \quad (M(D) \ge 0)$$

$$H_a: M(Y) < M(X) \quad (M(D) < 0)$$

Setting 3

$$H_0: M(Y) = M(X) \quad (M(D) = 0)$$

$$H_a: M(Y) \neq M(X) \quad (M(D) \neq 0)$$

Rank  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  according to the values of  $D_i \mid (i = 1, 2, \dots, n)$ .

$$R_i = \begin{cases} R\left(\left(X_i, Y_i\right)\right) & \text{if } Y_i > X_i \ \left(D_i > 0\right) \\ -R\left(\left(X_i, Y_i\right)\right) & \text{if } Y_i < X_i \ \left(D_i < 0\right) \end{cases}$$

Test statistics:

$$T^+ = \sum_{D_i > 0} R_i$$
 (without considering ties)

$$T = \frac{\sum_{i=1}^{n} R_i}{\sqrt{\sum_{i=1}^{n} R_i^2}}$$
 (considering ties)

Setting 1

$$H_0: M(Y) \le M(X) \quad (M(D) \le 0)$$

$$H_a: M(Y) > M(X) \quad (M(D) > 0)$$

Case A. There are no ties, and  $n \le 50$ .

Reject 
$$H_0$$
 if  $T_{\text{(obs)}}^+ > \frac{n(n+1)}{2} - w_{\alpha}$ .

The value of  $w_{\alpha}$  can be found from Table A12.

Case B. There are many ties, or n > 50.

Reject 
$$H_0$$
 if  $T_{\text{(obs)}} > z_{1-\alpha}$ .

$$p - \text{value} = P \left( Z \ge \frac{\sum_{i=1}^{n} R_i - 1}{\sqrt{\sum_{i=1}^{n} R_i^2}} \right)$$

Setting 2

$$H_0: M(Y) \ge M(X) \quad (M(D) \ge 0)$$

$$H_a: M(Y) < M(X) \quad (M(D) < 0)$$

Case A. There are no ties, and  $n \le 50$ .

Reject 
$$H_0$$
 if  $T_{\text{(obs)}}^+ < w_{\alpha}$ .

The value of  $w_{\alpha}$  can be found from Table A12.

Case B. There are many ties, or n > 50.

Reject  $H_0$  if  $T_{\text{(obs)}} < z_{\alpha}$ .

$$p\text{-value} = P \left( Z \le \frac{\sum_{i=1}^{n} R_i + 1}{\sqrt{\sum_{i=1}^{n} R_i^2}} \right)$$

Setting 3

$$H_0: M(Y) = M(X) \quad (M(D) = 0)$$

$$H_a: M(Y) \neq M(X) \quad (M(D) \neq 0)$$

Case A. There are no ties, and  $n \le 50$ .

Reject 
$$H_0$$
 if  $T_{\text{(obs)}}^+ < w_{\alpha/2}$  or  $T_{\text{(obs)}}^+ > \frac{n(n+1)}{2} - w_{\alpha/2}$ .

The value of  $w_{\alpha/2}$  can be found from Table A12.

Case B. There are many ties, or n > 50.

Reject 
$$H_0$$
 if  $T_{({\rm obs})} < z_{lpha/2}$  or  $T_{({\rm obs})} > z_{{\rm l}-lpha/2}$  .

$$p \text{-value} = 2 \min \left\{ P \left( Z \le \frac{\sum_{i=1}^{n} R_i + 1}{\sqrt{\sum_{i=1}^{n} R_i^2}} \right), P \left( Z \ge \frac{\sum_{i=1}^{n} R_i - 1}{\sqrt{\sum_{i=1}^{n} R_i^2}} \right) \right\}$$

#### EXAMPLE 1

Twelve sets of identical twins were given psychological tests to measure in some sense the amount of aggressiveness in each person's personality. We are interested in comparing the twins with each other to see if the firstborn twin tends to be more aggressive than the other. The results are as follows, where the higher score indicates more aggressiveness.

		Twin Set										
	1	2	3	4	5	6	7	8	9	10	11	12
Firstborn $X_i$	86	71	77	68	91	72	77	91	70	71	88	87
Second twin $Y_i$	88	77	76	64	96	72	65	90	65	80	81	72

 $H_0$ : The firstborn twin does not tend to be more aggressive than the other.

$$(M(Y) \ge M(X))$$

 ${\cal H}_a$  : The first born twin tends to be more aggressive than the second twin.

		Twin Set										
	1	2	3	4	5	6	7	8	9	10	11	12
Firstborn $X_i$	86	71	77	68	91	72	77	91	70	71	88	87
Second twin $Y_i$	88	77	76	64	96	72	65	90	65	80	81	72
Difference $D_i$	+2	+6	-1	-4	+5	0	-12	-1	-5	+9	-7	-15
Rank of $ D_i $	3	7	1.5	4	5.5	-	10	1.5	5.5	9	8	11
$R_i$	3	7	-1.5	-4	5.5	-	-10	-1.5	-5.5	9	-8	-11

• Use the Wilcoxon Signed Ranks Test for One Sample Problem

## EXAMPLE 2

Thirty observations on a random variable Y are obtained in order to test the hypothesis that M(Y), the median of Y, is no larger than 30. The data set is shown below.

$Y_{i}$	$Y_{i}$
23.8	35.9
26.0	36.1
26.9	36.4
27.4	36.6
28.0	37.2
30.3	37.3
30.7	37.9
31.2	38.2
31.3	39.6
32.8	40.6
33.2	41.1
33.9	42.3
34.3	42.8
34.9	44.0
35.0	45.8

 $H_0: M(Y) \le 30$   $H_a: M(Y) > 30$ 

$Y_{i}$	$D_i = Y_i - 30$	Rank of $ D_i $
23.8	-6.2	17
26.0	-4.0	11
26.9	-3.1	8
27.4	-2.6	6
28.0	-2.0	5
30.3	+0.3	1
30.7	+0.7	2
31.2	+1.2	3
31.3	+1.3	4
32.8	+2.8	7
33.2	+3.2	9
33.9	+3.9	10
34.3	+4.3	12
34.9	+4.9	13
35.0	+5.0	14

$Y_{i}$	$D_i = Y_i - 30$	Rank of $ D_i $
35.9	+5.9	15
36.1	+6.1	16
36.4	+6.4	18
36.6	+6.6	19
37.2	+7.2	20
37.3	+ 7.3	21
37.9	+7.9	22
38.2	+8.2	23
39.6	+9.6	24
40.6	+10.6	25
41.1	+11.1	26
42.3	+12.3	27
42.8	+12.8	28
44.0	+14.0	29
45.8	+15.8	30

· Confidence Interval for the Median Difference

Let  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  be a random sample from a bivariate population distribution (X, Y). The purpose is to find a confidence interval for the median difference M(Y) - M(X).

Step1. Define  $D_i = Y_i - X_i$  ( $i = 1, 2, \dots, n$ ).

Let  $D^{(1)}, D^{(2)}, \dots, D^{(n)}$  be the order statistics of  $D_1, D_2, \dots, D_n$ .

Step 2. Find  $w_{\alpha/2}$  from Table A12.

Step 3. Consider all possible averages  $A_{ij} = (D_i + D_j)/2$   $(i, j = 1, 2, \dots, n)$ .

The lower confidence limit is the  $w_{\alpha/2}$ th smallest  $A_{ii}$  value.

The upper confidence limit is the  $w_{\alpha/2}$ th largest  $A_{ii}$  value.

The inclusion of the end points is recommended by Moses (1965).

EXAMPLE 1 (continued)

The 12 values of D, arranged in order, are

$$-15$$
,  $-12$ ,  $-7$ ,  $-5$ ,  $-4$ ,  $-1$ ,  $-1$ ,  $0$ ,  $2$ ,  $5$ ,  $6$ ,  $9$ .

Find a 95% confidence interval for the median difference.

n = 12

$$W_{\alpha/2} = W_{0.025} = 14$$
 (Table A12)

# Section 5.8 Several related samples

## Randomized Complete Block Design

	Treatment									
Block	1	2		k						
1	$x_{11}$	x <sub>12</sub>	•••	$x_{1k}$						
2	$x_{21}$	x <sub>22</sub>		$x_{2k}$						
			•••							
b	$x_{b1}$	$x_{b2}$		$\mathcal{X}_{bk}$						

# • Friedman Test

The purpose of the test is to check if there is a difference among the treatment medians.

### Statistical test

 $\boldsymbol{H}_0$ : All  $\boldsymbol{k}$  treatments have the same effect on the response variable.

 ${\cal H}_a$  : At least two treatments have different effect on the response variable.

Let  $R(X_{ij})$  be the rank of  $X_{ij}$  within block i.

Case A. There are no ties.

$$T_{1} = \frac{12}{bk(k+1)} \sum_{j=1}^{k} \left( R_{j} - \frac{b(k+1)}{2} \right)^{2} = \frac{12}{bk(k+1)} \sum_{j=1}^{k} R_{j}^{2} - 3b(k+1)$$

$$\left( R_{j} = \sum_{i=1}^{k} R(X_{ij}) \right)$$

$$T_{2} = \frac{(b-1)T_{1}}{b(k-1) - T_{1}}$$

Reject  $H_0$  if  $T_{2(obs)} > F_{k-1,(b-1)(k-1),1-\alpha}$  (Table A22)

Case B. There are ties.

$$T_{1} = \frac{(k-1)\left(\sum_{j=1}^{k} R_{j}^{2} - b C_{1}\right)}{A_{1} - C_{1}} = \frac{(k-1)\sum_{j=1}^{k} \left(R_{j} - \frac{b(k+1)}{2}\right)^{2}}{A_{1} - C_{1}}$$
$$\left(A_{1} = \sum_{i=1}^{b} \sum_{j=1}^{k} \left[R\left(X_{ij}\right)\right]^{2} \quad C_{1} = \frac{bk(k+1)^{2}}{4}\right)$$

$$T_2 = \frac{(b-1)T_1}{b(k-1) - T_1}$$

Reject  $H_0$  if  $T_{2(\text{obs})} > F_{k-1,(b-1)(k-1),1-\alpha}$ . (Table A22)

## EXAMPLE (Reaction Time)

Suppose that three drugs A, B, and C, are to be compared using a randomized block design. Each of the three drugs is administrated to the same subject with suitable time lags among the three doses. Suppose six subjects are chosen and that the reaction times for each drug are shown in the following table.

Subject	Drug A	Drug B	Drug C
1	1.21	1.48	1.56
2	1.63	1.85	2.01
3	1.42	2.06	1.70
4	2.43	1.98	2.64
5	1.16	1.27	1.48
6	1.94	2.44	2.81

Is there sufficient evidence to indicate that three drugs are different? Use  $\alpha \text{=} 0.05.$ 

	Dru	ıg A	Drug B		Drug C	
Subject	Time	Rank	Time	Rank	Time	Rank
1	1.21	1	1.48	2	1.56	3
2	1.63	1	1.85	2	2.01	3
3	1.42	1	2.06	3	1.70	2
4	2.43	2	1.98	1	2.64	3
5	1.16	1	1.27	2	1.48	3
6	1.94	1	2.44	2	2.81	3
		$R_1 = 7$		$R_2 = 12$		$R_3 = 17$

#### EXAMPLE 1

Twelve homeowners are randomly selected to participate in an experiment with a plant nursery. Each homeowner was asked to select four fairly identical areas in his yard and to plant four different types of grasses, one in each area. At the end of a specified length of time each homeowner was asked to rank the grass types in order of preference, weighing important criteria such as expense, maintenance and upkeep required, beauty, hardiness, wife's preference, and so on. The rank 1 was assigned to the least preferred grass and the rank 4 to the favorite. Each of the 12 blocks consists of four fairly identical plots of land, each receiving care of approximately the same degree of skill because the four plots are presumably cared for by the same homeowner. The results of the experiment are as follows.

	Grass					
Homeowner	1	2	3	4		
1	4	3	2	1		
2	4	2	3	1		
3	3	1.5	1.5	4		
4	3	1	2	4		
5	4	2	1	3		
6	2	2	2	4		
7	1	3	2	4		
8	2	4	1	3		
9	3.5	1	2	3.5		
10	4	1	3	2		
11	4	2	3	1		
12	3.5	1	2	3.5		
$R_{j}$ (totals)	38	23.5	24.5	34		

## **Multiple Comparisons**

If the null hypothesis is rejected, then the following method can be used to compare treatments pair by pair.

Treatments i and j are considered different if the following inequality is satisfied:

$$|R_i - R_j| > t_{(b-1)(k-1),1-\alpha/2} \left[ \frac{2\left(bA_1 - \sum_{j=1}^k R_j^2\right)}{(b-1)(k-1)} \right]^{1/2}.$$

An equivalent expression is

$$\left| R_i - R_j \right| > t_{(b-1)(k-1), 1-\alpha/2} \left[ \frac{2b(A_1 - C_1)}{(b-1)(k-1)} \left( 1 - \frac{T_1}{b(k-1)} \right) \right]^{1/2}.$$

EXAMPLE 1 (Continued)

$$R_i$$
 (totals) 38 23.5 24.5 34

$$A_1 = \sum_{i=1}^b \sum_{j=1}^k \left[ R(X_{ij}) \right]^2$$

$$C_1 = \frac{bk(k+1)^2}{4}:$$

$$t_{(b-1)(k-1),1-\alpha/2} \left[ \frac{2 \left( b A_1 - \sum_{j=1}^k R_j^2 \right)}{(b-1)(k-1)} \right]^{1/2}$$

Pair	$ R_i - R_j $	> 11.49 ?
(1, 2)	14.5	yes
(1, 3)	13.5	yes
(1, 4)	4	no
(2, 3)	1	no
(2, 4)	10.5	no
(3, 4)	9.5	no

# Quade Test

Similar to the Friedman test, the Quade test uses the ranking of observations within blocks, but it gives more weight to the blocks with wider ranges.

Randomized Complete Block Design

	Treatment					
Block	1	2		k		
1	$x_{11}$	<i>x</i> <sub>12</sub>	•••	$\mathcal{X}_{1k}$		
2	$x_{21}$	$x_{22}$		$x_{2k}$		
b	$x_{b1}$	$x_{b2}$	•••	$\mathcal{X}_{bk}$		

## Statistical test

 $H_0$ : All k treatments have the same effect on the response variable.

 $H_a$ : At least two treatments have different effect on the response variable.

	Treatment					
Block	1	2		k		
1	<i>x</i> <sub>11</sub>	<i>x</i> <sub>12</sub>		$X_{1k}$		
2	$x_{21}$	x <sub>22</sub>		$x_{2k}$		
•••						
b	$x_{b1}$	$x_{b2}$		$x_{bk}$		

 $\text{range of block } i \ = \ \max_{j} \left\{ X_{ij} \right\} - \min_{j} \left\{ X_{ij} \right\} \ \left( i = 1, 2, \cdots, b \right)$ 

 $Q_i$   $(i = 1, 2, \dots, b)$ : ranks of the ranges

$$S_{\vec{y}} = Q_i \left[ R\left(X_{\vec{y}}\right) - \frac{k+1}{2} \right] \left(i = 1, 2, \cdots, b; j = 1, 2, \cdots, k\right)$$

Test statistic: 
$$T_3 = \frac{(b-1)B}{A_2 - B}$$

$$A_2 = \sum_{i=1}^b \sum_{j=1}^k S_{ij}^2 \quad \text{(total sum of squares)}$$

$$B = \frac{1}{b} \sum_{j=1}^k S_j^2 = \frac{1}{b} \sum_{j=1}^k \left( \sum_{i=1}^b S_{ij} \right)^2 \quad \text{(treatment sum of squares)}$$

If there are no ties, then  $A_2 = \frac{b(b+1)(2b+1)k(k+1)(k-1)}{72}$ .

Reject  $H_0$  if  $T_{3(obs)} > F_{k-1,(b-1)(k-1),1-\alpha}$  (Table A22)

$$p$$
-value =  $P(T_3 \ge T_{3(obs)})$ 

If  $A_2 = B$ , then  $H_0$  should be rejected because

(total sum of squares) = (treatment sum of squares)

means that all the variation is caused by the difference among the treatments.

The *p*-value in this case is  $(1/k!)^{b-1}$ .

#### **Multiple Comparisons**

Treatments i and j are considered to be different if

$$\left| S_i - S_j \right| > t_{(b-1)(k-1), 1-\alpha/2} \left[ \frac{2b(A_2 - B)}{(b-1)(k-1)} \right]^{1/2}.$$

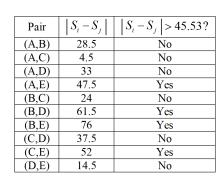
## EXAMPLE 2

Seven stores are selected for the marketing survey. In each store five different brands of a new type of hand lotion are placed side by side. At end of the week, the number of bottles of lotion sold for each brand is tabulated with the following results.

	Brand							
Store	A	В	C	D	Е			
1	5	4	7	10	12			
2	1	3	1	0	2			
3	16	12	22	22	35			
4	5	4	3	5	4			
5	10	9	7	13	10			
6	19	18	28	37	58			
7	10	7	6	8	7			

Are the five brands equally preferred by customers? Use  $\alpha = 0.05$ ?

					Brand		
Store	Range	$Q_{i}$	A	В	С	D	Е
1	8	5	5 (2) -5	4 (1) -10	7 (3) 0	10 (4) 5	12 (5) 10
2	3	2	1 (2.5) -1	3 (5) 4	1 (2.5) -1	0 (1) -4	2 (4) 2
3	23	6	16 (2) -6	12 (1) -12	22 (3.5) 3	22 (3.5) 3	35 (5) 12
4	2	1	5 (4.5) 1.5	4 (2.5) -0.5	3 (1) -2	5 (4.5) 1.5	4 (2.5) -0.5
5	6	4	10 (3.5) 2	9 (2) -4	7 (1) -8	13 (5) 8	10 (3.5) 2
6	40	7	19 (2) -7	18 (1) -14	28 (3) 0	37 (4) 7	58 (5) 14
7	4	3	10 (5) 6	7 (2.5) -1.5	6 (1) -6	8 (4) 3	7 (2.5) -1.5



$S_j$	-9.5	-38	-14	23.5	38

Summary of multiple comparisons: B C A D E

· Page Test for Ordered Alternatives

$$\begin{split} H_0 \colon \mu_{\mathbf{l}} &= \mu_{\mathbf{l}} = \mu_{\mathbf{l}} = \cdots = \mu_{\mathbf{k}} \quad \left( M_{\mathbf{l}} = M_{\mathbf{l}} = M_{\mathbf{3}} = \cdots = M_{\mathbf{k}} \right) \\ H_a \colon \mu_{\mathbf{l}} &\le \mu_{\mathbf{l}} \le \mu_{\mathbf{l}} \le \mu_{\mathbf{3}} \le \cdots \le \mu_{\mathbf{k}} \quad \left( M_{\mathbf{l}} \le M_{\mathbf{l}} \le M_{\mathbf{3}} \le \cdots \le M_{\mathbf{k}} \right) \end{split}$$

Let  $R(X_{ij})$  be the rank of  $X_{ij}$  within block i.

$$R_j = \sum_{i=1}^k R(X_{ij}) \ (j=1,2,\cdots,k) \ (R_j \text{ is the rank sum of the } j \text{ th treatment.})$$

$$\begin{split} R_j &= \sum_{i=1}^k R\left(X_{ij}\right) \; \left(j=1,2,\cdots,k\right) \quad \left(R_j \text{ is the rank sum of the } j \text{ th treatment.}\right) \\ \text{Test statistic:} \quad T_5 &= \frac{\sum_{j=1}^k j R_j - \frac{b k (k+1)^2}{4}}{\sqrt{\frac{b \left(k^3-k\right)^2}{144 \left(k-1\right)}}} \end{split}$$

Reject  $H_0$  if  $T_{5\, ({\rm obs})}>z_{1-\alpha}$  .

$$p$$
-value =  $P(Z \ge T_{5 \text{ (obs)}})$ 

## EXAMPLE 3

Health researchers suspect that regular exercise has a tendency to lower the pulse rate of a resting individual. To test this theory, eight healthy volunteers, who did not exercise on a regular basis, were enrolled in a supervised exercise program. The resting pulse rate was measured at the beginning of the program, and again after each month for four months. The observed pulse rates are shown in the table.

Person	Initial $(\mu_{\scriptscriptstyle 5})$	Month 1 $(\mu_4)$	Month 2 $(\mu_3)$	Month 3 $(\mu_2)$	Month 4 ( $\mu_1$ )
1	82	84	77	76	79
2	80	80	76	76	78
3	75	78	77	74	72
4	65	72	68	65	66
5	77	74	72	75	75
6	68	69	65	66	64
7	70	74	68	70	68
8	77	76	78	72	70

$$H_0$$
:  $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$   $H_a$ :  $\mu_1 \le \mu_2 \le \mu_3 \le \mu_4 \le \mu_5$ 

Section 5.9 The Balanced Incomplete Block Design

### Incomplete Block Design

(Some  $X_{ij}$  values are missing)

	Treatment					
Block	1	2		t		
1	<i>x</i> <sub>11</sub>	<i>x</i> <sub>12</sub>		$x_{1t}$		
2	$x_{21}$	x <sub>22</sub>		$x_{2t}$		
b	$x_{b1}$	$x_{b2}$		$x_{bt}$		

A design is called a balanced incomplete block design if the following conditions hold.

- 1. Some  $x_{ij}$  values are missing.
- 2. Every block contains k experimental units.
- 3. Every treatment appears in r blocks.
- 4. Every treatment appears with every other treatment an equal number of times ( $\lambda$  times).

### • Durbin Test

The purpose of the test is to check if there is a difference among the treatment medians.

#### Statistical test

 $H_0$ : All t treatments have the same effect on the response variable.

 $H_a$ : At least two treatments have different effect on the response variable.

Let  $R(X_{ij})$  be the rank of  $X_{ij}$  (if exists) among the available  $X_{ij}$ 's within block i.

Denote 
$$R_j = \sum_{i=1}^{b} R(X_{ij}) (j = 1, 2, \dots, t).$$

Test statistic

Case A. There are no ties.

$$T_{1} = \frac{12(t-1)}{rt(k-1)(k+1)} \sum_{j=1}^{t} \left( R_{j} - \frac{r(k+1)}{2} \right)^{2} \qquad \left( R_{j} = \sum_{i=1}^{b} R(X_{ij}) \right)$$

$$T_2 = \frac{T_1/(t-1)}{(b(k-1)-T_1)/(bk-b-t+1)}$$

Reject  $H_0$  if  $T_{2(\text{obs})} > F_{t-1,bk-b-t+1,1-\alpha}$ . (Table A22)

Case B. There are ties.

$$T_{1} = \frac{(t-1)\sum_{j=1}^{t} \left(R_{j} - \frac{r(k+1)}{2}\right)^{2}}{A - C} = \frac{(t-1)\left(\sum_{j=1}^{t} R_{j}^{2} - rC\right)}{A - C}$$

$$\left(A = \sum_{i=1}^{b} \sum_{j=1}^{t} \left[R(X_{ij})\right]^{2} \quad C = \frac{bk(k+1)^{2}}{4}\right)$$

$$T_{2} = \frac{T_{1}/(t-1)}{\left(b(k-1) - T_{1}\right)/\left(bk - b - t + 1\right)}$$

Reject  $H_0$  if  $T_{2(\text{obs})} > F_{k-1,bk-b-t+1,1-\alpha}$ . (Table A22)

# EXAMPLE 1

Suppose an icecream manufacturer wants to test the taste preferences of several people for her seven variaties ice cream. She asks each person to taste three variaties and rank them 1, 2, and 3, with the rank 1 being assigned to the favorite variety. In order to design the experiment so that each variety is compared with every other variety an equal number of times, a Youden square layout given by Federer (1963) is used. Seven people are each given three variaties, and the resulting ranks are as follows:

				Variaty			
Person	1	2	3	4	5	6	7
1	2	3		1			
2		3	1		2		
3			2	1		3	
4				1	2		3
5	3				1	2	
6		3				1	2
7	3		1				2

	Variaty						
Person	1	2	3	4	5	6	7
1	2	3		1			
2		3	1		2		
3			2	1		3	
4				1	2		3
5	3				1	2	
6		3				1	2
7	3		1				2
	$R_{1} = 8$	$R_2 = 9$	$R_{_3} = 4$	$R_4 = 3$	$R_{\scriptscriptstyle 5} = 5$	$R_6 = 6$	$R_7 = 7$

## Multiple Comparisons

If the null hypothesis is rejected, then the following method can be used to compare treatments pair by pair.

Treatments i and j are considered different if the following inequality is satisfied:

$$\left|\left.R_i-R_j\right.\right|>t_{bk-b-t+1,1-\alpha/2}\left\lceil\frac{2\,r\left(A-C\right)}{bk-b-k+1}\left(1-\frac{T_1}{b\left(k-1\right)}\right)\right\rceil^{1/2}.$$

If there are no ties, the expression can be simplified as

$$\left|R_i-R_j\right|>t_{bk-b-k+1,1-\alpha/2}\left[\frac{rk(k+1)}{6(bk-b-k+1)}\left(b(k-1)-T_1\right)\right]^{1/2}.$$

EXAMPLE 1 (Continue)

$$t_{bk-b-k+1,1-\alpha/2} \left\lceil \frac{rk(k+1)}{6(bk-b-k+1)} \left(b(k-1)-T_1\right) \right\rceil^{1/2}$$

Pair	$R_i - R_j$	> 2.2780 ?
(1,2)	1	No
(1,3)	4	Yes
(1,4)	5	Yes
(1,5)	3 2 1	Yes
(1,6)	2	No
(1,7)	1	No
(2,3)	5	Yes
(2,4) (2,5)	6	Yes
(2,5)	4	Yes
(2,6)	3 2	Yes
(2,7) (3,4)		No
(3,4)	1	No
(3,5)	1	No
(3,6)	2	No
(3,7)	1 2 3 2	Yes
(4,5)	2	No
(4,6)	3	Yes
(4,7)	4	Yes
(5,6)	1	No
(5,7)	2	No
(6,7)	1	No

Summary of multiple comparisons:



Variaty 4 is preferred over Variaties 6, 7, 1, and 2.

Variaty 3 is preferred over Variaties 7, 1, and 2.

Variaty 5 is preferred over Variaties 1, and 2.

Variaty 6 is preferred over Variaty 2.